axis is stabilized at attitude $A5$ until the end of the simulation. The following parameters were assumed in the simulation: $m_r = -0.1\text{Am}^2$, $m_i = 15\text{Am}^2$, $m_p = 0.5\text{Am}^2$, $p = 1925.89$ m^4/Ω, $I_z = 10$ kgm^2, $I_x = 9.8$ kgm^2, altitude 750 km (circular orbit), inclination 25 deg.

Figure 1 illustrates both the satellite attitude behavior and error angle $2\sin^{-1}(E/2)$ during the reorientation maneuver. Figure 2 is a plot of the control variable u as a function of time during approximately three orbits. The effect of the spin rate control is shown in Fig. 3. The nominal spin rate has been taken as 10 rpm and the maximum error allowable is 0.1 rpm. Since the attitude dynamics is slow with respect to the spin period, the mean value of the plane magnetic coil torque in one period has been used in the integration of $\dot{\omega}$ [see Eq. (1)].

Conclusions

An example of the practical application of the ideas contained in Ref. 3 to the reorientation of the spin axis as well as the spin rate control of the BDCS has been presented. The simulation results suggest the feasibility of performing the reorientation maneuvers by ground command; furthermore, the spin rate control can be done by an onboard computer using magnetometer phase signals to properly switch the plane coil.

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Approach to Robust Control Systems Design

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Introduction

THE problem of robustness has received considerable attention in recent years. Some of the recent research in this area was inspired by the Kharitonov theorem related to the asymptotic stability of a family of systems described by linear differential equations.[1] Different generalizations of the Kharitonov theorem have been discussed in Refs. 2 and 3.

Based on the criteria similar to the well-known ones in the classical automatic control theory, the problem of robust stability in linear time-invariant closed-loop systems with parametric uncertainty is formulated mostly, in essence, as the analysis problem. Using classical techniques, the stability of the system under consideration can be examined and the controller parameters, which guarantee the asymptotic stability of the closed-loop system, can be determined. Usually, it is assumed that a controller has been constructed for the so-called nominal plant. By using the Kharitonov-type theorem, one can only check if the system with perturbed parameters remains stable. No effective design procedure exists for changing the controller parameters to make the real (not nominal) system asymptotically stable.

Parallel with the approach just discussed, the Lyapunov-Bellmen method is used to design controllers for a system with uncertain dynamics (see, e.g., Refs. 4 and 5).

This paper presents an approach to the synthesis of robust control systems with uncertain parameters which is based on the consideration of an optimal control problem with a specified performance index. The introduced estimate of the location of the eigenvalues of the plant with uncertain parameters allows us to formulate the optimal control problem, the solution of which guarantees the asymptotic stability of the closed-loop system. The main advantage of the proposed procedure is its simplicity. It is similar to the well-known procedure of the analytical controller design based on the solution of the linear quadratic optimal control problem with the integral quadratic performance index.

Statement of the Problem and Main Results

Let us consider the linear controllable plant described by the equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where x is an n-dimensional state-space vector, u is an m-dimensional control vector, and $A = |a_{ij}|$ and $B = |b_{ij}|$ are matrices of appropriate dimensions.

It is assumed first that only elements of the state matrix A are not known exactly, i.e.,

$$A_1 \leq A \leq A_2 \tag{2}$$

where $A_1 = |a_{1ij}|$ and $A_2 = |a_{2ij}|$ characterize the upper and lower bounds of A, respectively.

The robust control problem consists of finding controller equations that make the closed-loop system asymptotically stable for all state matrices of the form of Eq. (2).

The well-known procedure of analytical controller design for systems of the form of Eq. (1) is based on minimization of the functional

$$J_0 = \frac{1}{2}\int_0^\infty [x^T(t)Qx(t) + cu^T(t)u(t)]\,dt \tag{3}$$

where $Q = |q_{ij}|$ is a non-negative definite symmetric matrix, and c is a positive constant.

The control law has the form

$$u(t) = -\frac{1}{c}B^TWx(t) \tag{4}$$

where the positive-definite matrix W satisfies the Riccati equation.

Unlike the standard procedure of the analytical controller design, in the case under consideration we lack the proper information to calculate the matrix W.

It is known[6] that minimization of the functional

$$J_1 = \frac{1}{2}\int_0^\infty e^{2\gamma t}[x^T(t)Qx(t) + cu^T(t)u(t)]\,dt \tag{5}$$

subject to the boundary conditions described by system (1) is equivalent to minimization of the functional (3) subject to the

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system

$$\dot{x}(t) = A_0 x(t) + Bu(t) \tag{6}$$

$$A_0 = A + \gamma I \tag{7}$$

where the eigenvalues of the matrix A_0 are shifted by γ in comparison to the eigenvalues of the matrix A.

A special optimal control problem for system (1) with fixed constant parameters is considered. Some information regarding choosing γ in Eq. (5) can be obtained from the estimate of the upper bound of the eigenvalues of the family of state matrices (2). We will call γ the upper bound of the eigenvalues of the matrices A of Eq. (1) if the half-plane $Re\, s \geq \gamma$ contains no eigenvalues of family (2) of A.

A simple estimate of γ follows from the expressions (see the Appendix)

$$\gamma = \max\,[1,\ (n+\epsilon)\max_{i,a_{ij}} |M_i|\,] \tag{8a}$$

$$\gamma = \min[\max_{j,a_{ij}} \sum_{j=1}^{n} |a_{ij}|,\quad \max_{j,a_{ij}} \sum_{i=1}^{n} |a_{ij}|\,] \tag{8b}$$

$$\gamma = \tfrac{1}{2}\, n \max_{i,j,a_{ij}} |a_{ij} + a_{ji}| \tag{8c}$$

where M_i denotes the sum of all the principle minors of order i, $1 \leq i \leq n$; ϵ is a small positive number.

Theorem 1. Given a controllable system [Eq. (1)] with the elements of the state matrix satisfying inequalities (2), then the optimal control law [Eq. (4)], which minimizes the cost functional (3) [or (5)] subject to system (1) with a certain matrix A_*, makes Eqs. (1) and (4) asymptotically stable for all matrices A satisfying inequalities (2).

Proof. For an arbitrary matrix A, the closed-loop system described by Eqs. (1) and (4) is asymptotically stable for a positive-definite W, which is obtained from the solution of the optimal problem of Eqs. (1) and (3). Let A_* correspond to the "worst" case of A, i.e., to the case of a positive-definite $W_* > W$. Then, by considering the derivative of $V = \tfrac{1}{2} x^T(t) W x(t)$, along with Eq. (1) with an arbitrary A and Eq. (4) with $W = W_*$, and taking into account that $W_* - W$ is a positive-definite matrix, we obtain $dV/dt < 0$. In the case where there exists no A_* among the family (2), or it is difficult to find it, an extended domain can be built in which we can find such a matrix. One of the simplest ways to build A_* consists of shifting the eigenvalues of the state matrix A. Hence, by solving the optimal synthesis problem of Eqs. (1) and (5) for the indicated matrix A_*, we obtain the controller equation [Eq. (4)], which guarantees the asymptotic stability of systems (1) and (4) for all matrices of the given type [Eq. (2)].

Now we assume that the elements of the input matrix B belong to a domain that is described by

$$B_1 \leq B \leq B_2 \tag{9}$$

where, unlike Eq. (2), the upper and lower bounds of b_{ij} are assumed to have the same sign.

Such a condition corresponds to the realistic case when an investigator knows qualitatively (but not quantitatively) how the controls influence the state variables.

Theorem 2. Let the control [Eq. (4)] be determined for the functional [Eq. (3)], a certain state matrix A_*, and the input matrix equal to B_1. Then the closed-loop systems (1) and (4) are asymptotically stable for all of the matrices A and B of the form of Eqs. (2) and (9), respectively.

This theorem can be considered as a part of the more general theorem that will be given later.

Let us consider the nonlinear controllable plant of the type

$$\dot{x}(t) = Ax(t) + Bg[u(t)] \tag{10}$$

where A and B are matrices of the form of Eqs. (2) and (9); $g(u) = [g(u_1),\ldots,g(u_m)]$ is a vector function whose elements satisfy the conditions

$$h_{ii}^0 u_i^2 \leq u_i g_i(u_i), \qquad h_{ii}^0 > 0 \qquad (i = 1,\ldots, m) \tag{11}$$

[It is assumed that $g(u)$ is smooth enough to guarantee the existence of the solution of Eq. (10) for any given initial conditions and controls u.]

Theorem 3. Let there exist the optimal solution (4) of the problems (1) and (3) for a certain state matrix A_*, and $B = B_1 H$, $H^0 - H \geq 0$, $H = |h_{ii}| > 0$, and $H^0 = |h_{ii}^0| > 0$. Then the nonlinear systems (4) and (10) are absolutely stable for all A and B of the form of Eqs. (2) and (9), respectively.

For any fixed A and B the control (4) of system (1) with the input matrix BH, which is optimal with respect to the performance index (3) [or the performance index (5)], makes the nonlinear system (10) asymptotically stable.[7]

In the case of B belonging to the domain (9), we can describe the term $Bg(u)$ by introducing a set of nonlinear functions $\bar{g}(u)$ satisfying Eq. (11) and the condition $Bg(u) = B_1\bar{g}(u)$. This is due to the fact that only the lower bound is used in Eq. (11).

Hence, instead of system (10) with the family of input matrices B and the given $g(u)$, we can examine the nonlinear system with the constant matrix B_1:

$$\dot{x}(t) = Ax(t) + B_1\bar{g}(u) \tag{12}$$

where the vector function $\bar{g}(u)$ satisfies conditions (11).

According to Theorem 3, systems (10) and (4) under the control determined for $B = B_1 H$ are asymptotically stable.

Finally, according to Theorem 1, the optimal control (4) determined for system (1) with a certain A and $B = B_1 H$ makes systems (1) and (4) asymptotically stable for the entire family of A and B.

Conclusions

The proposed approach of designing a class of robust control systems is based on the consideration of the special optimal control problem for the system with specified constant parameters. The estimate of the upper bound of the eigenvalues of the family of the state matrices can be used to determine the exponential factor used in the performance index. The given procedure allows us to build robust linear systems as well as a wide class of robust nonlinear systems.

Appendix

The characteristic polynomial $f(s)$ of system (1) has the following form:

$$f(s) = s^n + P_1 s^{n-1} + \ldots + P_n(s)$$

where

$$P_i = (-1)^i M_i$$

and M_i is the sum of all the principal minors of order i of the matrix A.

The coefficients of $f(s)$ belong to domains that are defined by the elements of A in Eq. (1). The location of the roots of the corresponding characteristic equation (the eigenvalues of A) depends on the size of these domains. We will determine the domain of the complex plane $Re\, s \geq 0$, which does not contain the eigenvalues of the matrix A for all possible a_{ij} satisfying Eq. (2).

For a polynomial function of the complex variable $f(s)$ we have

$$f(s) \geq |s|^n [1 - \sum_{i=1}^{n} |P_i|\, |s|^{-i}]$$

$$\geq |s|^n [1 - \sum_{i=1}^{n} k_{\max}\, |s|^{-i}]$$

where

$$k_{\max} = \max_{i,a_{ij}} |M_i|$$

Assuming $|s| \geq 1$ we have

$$|f(s)| \geq |s|^n \left[1 - \frac{nk_{\max}}{|s|} \right] = |s|^{n-1}[|s| - nk_{\max}]$$

Hence,

$$|f(s)| > 0 \qquad \text{if} \qquad |s| \geq \max[1, (n + \epsilon)\, k_{\max}]$$

The upper bound of the eigenvalues of A follows immediately from this expression. The other two inequalities [Eqs. (8)] were, in essence, obtained by Hirsh and Brauer.[8]

It should be mentioned that, in practice, we estimate $k_{\max}$ approximately by determining its upper bound.

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